

FIRST EIGENVALUE ESTIMATES FOR THE p -LAPLACIAN ON COMPLETE RIEMANNIAN MANIFOLDS VIA CRITICAL SETS OF HARMONIC FUNCTIONS

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ABSTRACT

The nonlinear p -Laplacian operator plays a fundamental role in geometric analysis, nonlinear spectral theory, and elliptic partial differential equations. Estimating its first eigenvalue is of particular importance because it reflects both the analytical and geometric properties of the underlying Riemannian manifold. This paper proposes a variational framework for studying the first eigenvalue of the p -Laplacian through the geometry of critical sets of harmonic functions. The manuscript presents the mathematical formulation, theoretical background, and analytical framework that can be used to derive new eigenvalue estimates under suitable curvature assumptions. The relationships among variational methods, Sobolev inequalities, comparison geometry, and nonlinear elliptic equations are discussed, together with possible directions for original theorem development.

Keywords: p -Laplacian¹, First eigenvalue², Riemannian manifold³, Harmonic functions⁴, Critical sets⁵, Elliptic PDEs⁶.

1. INTRODUCTION

The study of nonlinear eigenvalue problems associated with the p -Laplacian has become one of the principal areas of nonlinear geometric analysis. Unlike the classical Laplace–Beltrami operator, the p -Laplacian introduces nonlinear behaviour through the exponent (p), making its analytical treatment substantially more challenging. The first eigenvalue is closely related to manifold geometry, stability of solutions, nonlinear diffusion, and variational principles.

Recent work has extended eigenvalue estimates to weighted manifolds, metric measure spaces, and nonlinear operators, highlighting the importance of curvature, Sobolev inequalities, and comparison geometry in spectral analysis.

2. PROBLEM STATEMENT

This study investigates the nonlinear eigenvalue problem

$$[-\Delta_p u = \lambda |u|^{p-2} u]$$

on a complete Riemannian manifold (M) , with the objective of relating the first eigenvalue to geometric properties of harmonic function critical sets.

3. OBJECTIVES

- To formulate the first eigenvalue problem variationally.
- To analyse the influence of Ricci curvature.
- To study critical sets of harmonic functions.
- To establish inequalities relating geometry and spectrum.
- To investigate implications for nonlinear elliptic PDEs.

4. MATHEMATICAL FRAMEWORK

The analysis is carried out in Sobolev spaces $(W^{1,p}(M))$.

The associated energy functional is

$$[J(u) = \frac{1}{p} \int_M |\nabla u|^p, dV]$$

Critical points satisfy

$$[\Delta_p u = 0]$$

The first eigenvalue is defined through the Rayleigh quotient

$$\left[\lambda_1 = \inf_{u \neq 0} \frac{\int_M |\nabla u|^p, dV}{\int_M |u|^p, dV} \right]$$

5. PROPOSED ANALYTICAL METHODOLOGY

The investigation combines:

- Variational methods.
- Sobolev embedding theorems.
- Bochner-type identities.
- Hölder and Poincaré inequalities.
- Comparison geometry.
- Maximum principles.
- Gradient estimates.

These tools are expected to provide estimates for the first eigenvalue under suitable geometric assumptions.

6. PRELIMINARY RESULTS

This section presents several preliminary results that form the analytical foundation for studying the first eigenvalue of the p -Laplacian on complete Riemannian manifolds. These propositions are standard consequences of variational methods, Sobolev space theory, and nonlinear elliptic partial differential equations. They are included to establish the notation and analytical framework used in the subsequent development.

Theorem 1 (Variational Characterization of the First Eigenvalue)

Let $((M, g))$ be a compact connected Riemannian manifold with smooth boundary (∂M) , and let $(1 < p < \infty)$. Then the first eigenvalue of the p -Laplacian is characterized by

$$\left[\lambda_{1,p} \inf_{u \in W_0^{1,p}(M) \setminus \{0\}} \frac{\int_M |\nabla u|^p, dV}{\int_M |u|^p, dV} \right]$$

Moreover, the infimum is attained by a nontrivial eigenfunction.

Proof

Consider the energy functional

$$\left[J(u) = \int_M |\nabla u|^p, dV \right]$$

Restrict the minimization to the constraint

$$\left[\int_M |u|^p, dV = 1. \right]$$

Since $(W_0^{1,p}(M))$ is reflexive and the functional (J) is convex, coercive, and weakly lower semicontinuous, the Direct Method of the Calculus of Variations guarantees the existence of a minimizer (u_0) .

Introducing a Lagrange multiplier (λ), the Euler–Lagrange equation becomes

$$[-\Delta_p u \lambda |u|^{p-2} u,]$$

which is precisely the nonlinear eigenvalue problem. Therefore the minimum value equals the first eigenvalue.

Theorem 2 (Positivity of the First Eigenvalue)

Let (M) be compact with nonempty boundary.

Then

$$[\lambda_{1,p} > 0.]$$

Proof

Suppose

$$[\lambda_{1,p} = 0.]$$

Then

$$\left[\int_M |\nabla u|^p, dV = 0, \right]$$

which implies

$$[\nabla u = 0]$$

almost everywhere.

Hence (u) is constant.

Since

$$[u \in W_0^{1,p}(M),]$$

the boundary condition forces

$$[u=0,]$$

contradicting the assumption that the eigenfunction is nontrivial.

Therefore

$$[\lambda_{1,p} > 0.]$$

Proposition 1 (Weak Formulation)

Suppose

$$[u \in W^{1,p}(M)]$$

is a weak solution of

$$[-\Delta_p u = f.]$$

Then

$$\left[\int_M |\nabla u|^{p-2} \langle \nabla u, \nabla \phi \rangle, dV \int_M f \phi, dV \right]$$

for every

$$[\phi \in C_0^\infty(M)]$$

Proof

Multiply the differential equation by the test function (ϕ) .

Integrate over the manifold.

Applying the divergence theorem yields

$$\left[\int_M \operatorname{div}(|\nabla u|^{p-2} \nabla u) \phi \int_M |\nabla u|^{p-2} \langle \nabla u, \nabla \phi \rangle \right]$$

Substituting the governing equation gives the required weak formulation.

Proposition 2 (Energy Estimate)

Let

$$[u \in W_0^{1,p}(M)]$$

solve

$$[-\Delta_p u = f]$$

Then

$$\left[\int_M |\nabla u|^p \leq C \int_M |f|^{p'} \right]$$

where

$$\left[\frac{1}{p} + \frac{1}{p'} = 1 \right]$$

Proof

Choosing

$$[\phi = u]$$

in the weak formulation,

$$\left[\int_M |\nabla u|^p \int_M f u \right]$$

Applying Hölder's inequality,

$$\left[\left| \int_M f u \right| \leq |f|_{L^{p'}} |u|_{L^p} \right]$$

Finally, using the Poincaré inequality,

$$[|u|_{L^p} \leq C |\nabla u|_{L^p}]$$

produces the stated estimate.

Proposition 3 (Existence of Weak Solutions)

Assume

$$[f \in L^{p'}(M)]$$

Then there exists at least one weak solution of

$$[-\Delta_p u = f]$$

Proof

Define

$$\left[I(u) \frac{1}{p} \int_M |\nabla u|^p \int_M f u \right]$$

The functional is coercive because

$$[I(u) \rightarrow \infty]$$

as

$$[|u|_{W^{1,p}} \rightarrow \infty]$$

Since it is convex and weakly lower semicontinuous, the Direct Method yields a minimizer.

Differentiating the functional gives the Euler–Lagrange equation,

$$[-\Delta_p u = f]$$

which establishes existence.

7. APPLICATIONS

Potential applications include:

- Nonlinear diffusion equations.
- Geometric flows.
- Stability of nonlinear elliptic equations.
- Mathematical physics.
- Spectral geometry.
- Nonlinear optimization.

8. DISCUSSION

The proposed framework illustrates how geometric information encoded in critical sets of harmonic functions may contribute to understanding nonlinear spectral properties of the p -Laplacian. Recent advances in regularity

theory, weighted p -Laplacians, and nonlinear eigenvalue problems suggest several promising avenues for extending classical eigenvalue estimates to broader geometric settings.

9. CONCLUSION

The paper outlines a research framework connecting first eigenvalue estimates, harmonic function critical sets, and nonlinear elliptic PDEs on Riemannian manifolds. The central mathematical contribution of the eventual research paper would consist of new inequalities and proofs establishing relationships between geometric invariants and the first eigenvalue of the p -Laplacian. Such results would contribute to nonlinear spectral geometry and provide analytical tools applicable to broader classes of elliptic problems.

10. REFERENCES:

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